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The O(N) model on a squashed S^3 and the Klebanov-Polyakov correspondence

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Abstract: We solve the O(N) vector model at large N on a squashed three-sphere with a conformal mass term. Using the Klebanov-Polyakov version of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence we match various aspects of the strongly coupled theory with the physics of the bulk AdS Taub-NUT and AdS Taub-Bolt geometries. Remarkably, we find that the field theory reproduces the behaviour of the bulk free energy as a function of the squashing parameter. The O(N) model is realised in a symmetric phase for all finite values of the coupling and squashing parameter, including when the boundary scalar curvature is negative.

Keywords: Field Theories in Lower Dimensions, 1/N Expansion, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction and summary

The AdS/CFT correspondence posits the existence of exact dualities between large N field theories and string theory on asymptotically anti-de Sitter (AdS) spacetimes [1–3]. As is well known, these gauge-gravity dualities typically relate two complex theories, each being tractable only in some limiting region of parameter space. A somewhat simpler version of such a duality was recently proposed by Klebanov and Polyakov [4] wherein the large N field theory is exactly solvable. Specifically, the Klebanov-Polyakov correspondence conjectures a duality between a theory of massless higher spin gauge fields in AdS_4 spacetime on the one hand, and the singlet sector of the critical O(N) vector model at large N in three dimensions on the other.

Motivated by the Klebanov-Polyakov proposal, in this paper we will solve the O(N) vector model at large N on homogeneous, compact manifolds known as squashed three-spheres. The squashed three-spheres are one parameter deformations of the round three-sphere which introduce an anisotropy whilst preserving homogeneity. Importantly, they are the conformal boundary of certain asymptotically locally anti-de Sitter spacetimes known as AdS Taub-NUT and, for certain values of the squashing parameter, of another geometry called AdS Taub-Bolt [5, 6]. Both these backgrounds are Einstein manifolds with constant negative curvature. The holographic correspondence then suggests an equivalence between higher spin gauge theory on the AdS Taub-NUT/Bolt geometries, and the O(N) vector model at large N on the boundary geometry.

These backgrounds were studied in the context of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence in [5−11] . However, in those works little could be said about the dual field theory because in the $AdS_4 \times S^7$ version of the gauge-gravity correspondence [1] the field theory is a strongly coupled conformal field theory which remains largely unknown. This is no longer the case for the Klebanov-Polyakov version of the correspondence, where the field theory is tractable at all values of the coupling, weak and strong.

We will study the O(N) theory on a squashed three-sphere as a function of the squashing parameter and a dimensionless coupling constant. In the absence of a complete formulation of the theory of higher spin gauge fields on the bulk geometries of interest, we compare exact results from field theory with semiclassical properties of Einstein gravity in the bulk [5−7, 10]. One of the main results of our study is a remarkable qualitative agreement between the free energy of the field theory at strong coupling and that of the bulk gravitational theory, as a function of the squashing parameter. This result is encapsulated in figures 3 and 4 below.

The squashed sphere is an S^1 bundle over S^2 where the squashing parameter α is related to the periodicity of the S^1 fibre. The periodicity of the S^1 fibre is often interpreted as the inverse of a temperature, giving $T \sim \sqrt{\alpha}$ for large α. This then allows a physical understanding of the fact that the bulk gravity theory with a squashed three-sphere as boundary exhibits a first order phase transition of Hawking-Page type [12] as this temperature is increased beyond a critical value [5, 6]. In particular, for fixed $\alpha < \alpha_{\mathrm{crit}}$ the semiclassical quantum gravity partition function is dominated by the AdS Taub-NUT geometry. Beyond α_{crit} however, the partition function is dominated by the AdS Taub-Bolt background. The AdS Taub-Bolt geometry should be thought of as a Euclidean black hole with a NUT charge.

Our field theory result for the free energy of the O(N) vector model at strong coupling will agree with the behaviour of AdS Taub-NUT for $\alpha < \alpha_{\mathrm{crit}}$ and with that of AdS Taub-Bolt at large values of α, but with a smooth crossover between the two regimes instead of a first order phase transition. We view the qualitative agreement away from $\alpha = \alpha_{\mathrm{crit}}$ as a positive test of the proposed holographic duality. We will discuss the possibility that accounting for the infinite massless higher spin degrees of freedom in the bulk will smooth out the first order Hawking-Page transition.

It is not surprising that the nature of the Hawking-Page transition here is qualitatively different to the cases in which it is dual to confinement/deconfinement transitions in large N Yang-Mills theory [2, 13−15]. These transitions are accompanied by a drastic change in the number of degrees of freedom of the boundary field theory. The natural phase transition one might expect in the O(N) model is a symmetry breaking transition as one

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varies α. However, we will show explicitly that the large N limit of the $\mathrm{O}(N)$ model on a compact space is always realized in an $\mathrm{O}(N)$-symmetric phase at finite coupling, with any possibility of spontaneous symmetry breaking of $\mathrm{O}(N) \to \mathrm{O}(N-1)$ precluded by finite volume effects.

Let us now briefly summarise a few further features and implications thereof of our field theoretic results. In flat space, the critical $\mathrm{O}(N)$ vector model in three dimensions exhibits a renormalisation group flow from a free fixed point in the ultraviolet (UV) to an interacting fixed point in the infrared (IR), see for example [4, 16, 17]. This flow is induced by the quartic interaction which is a relevant operator with a coupling constant that has mass dimension one. A compact space such as the squashed sphere introduces a second scale, thus providing a dimensionless coupling constant. The large N limit leads to an exactly solvable theory for any value of this dimensionless coupling constant.

We consider the theory with a conformal mass term so that in the ultraviolet the theory approaches a free conformal fixed point, presumably corresponding to the asymptotically AdS region of the bulk. Interestingly, the conformal mass term becomes negative for a range of values of the squashing, as the scalar curvature of the squashed sphere becomes negative. One of our results will be that nonetheless the theory is always realised in a symmetric phase for all finite values of the coupling and squashing.

In addition to matching the behaviour of the free energy with squashing as we have described above, we make two further connections between the physics of the $\mathrm{O}(N)$ model and the bulk theory

- A negative curvature of the boundary is often associated with instabilities in the bulk geometry, as reviewed in [10]. Our field theory results suggest that there is no instability of the classical bulk AdS Taub-NUT geometry in the higher spin gauge theory when the scalar curvature of the conformal boundary becomes negative. We will perform a check of this statement by considering fluctuations of a conformally coupled scalar field about the bulk background.

- We calculate the condensate $\langle\vec{\Phi}\cdot\vec{\Phi}\rangle$ in the $\mathrm{O}(N)$ model. This vacuum expectation value is dual, in the Klebanov-Polyakov correspondence, to the normalisable mode of a conformally coupled scalar field in the bulk, φ. This scalar field is not turned on in the AdS Taub-NUT or AdS Taub-Bolt geometries. Consistent with this fact we find $\langle\vec{\Phi}\cdot\vec{\Phi}\rangle \to 0$ at strong coupling. For boundaries other than the round sphere, this condensate does not vanish at finite coupling, predicting curvature corrections to the bulk AdS Taub-NUT/Bolt geometries, whilst AdS itself is protected from corrections.

Given that the nonperturbative large N resummation can be implemented in field theory for arbitrary values of the dimensionless coupling, we can in fact analyse the theory both at arbitrarily weak coupling and at infinitely strong coupling. In these two limits we find analytically the following field theory physics

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- The theory at arbitrarily weak dimensionless coupling, $a\lambda \to 0^+$, exhibits a phase transition at a value of the squashing parameter where the boundary curvature turns negative. The order parameter for this transition is the condensate $\lambda\langle\vec{\Phi}\cdot\vec{\Phi}\rangle$ which vanishes for negative curvature while remaining nonzero and of order N at positive curvatures. Interestingly, for any finite value of the coupling this phase transition gets completely smoothed out, as we stated above. Such nonperturbative effects may be relevant more generally for attempts to extrapolate perturbative phase transitions to strong coupling [13, 14].
- At large squashing parameter the free energy of the AdS Taub-Bolt geometries is known to depend linearly on the squashing parameter [5, 6]. We find that the O(N) model exhibits the same linear dependence at both strong and weak coupling. The ratio of the free energy at large squashing at weak and strong coupling is 4/5, rather analogously to the well-known 3/4 factor [18] for $\mathcal{N} = 4$ super Yang Mills, except that here we can calculate both limits in field theory.

The physics of interacting quantum fields on a squashed three-sphere received attention at various points in the eighties following the growth of interest in quantum fields on curved backgrounds. These works included considerations of quantum induced symmetry breaking through calculation of the one loop effective potential [19 − 23]. However, although one loop effective potentials can be useful, they are insufficient to understand the phase structure of field theories. Firstly because when quantum corrections become of the same order as the classical mass term then typically perturbation theory is not reliable. Secondly, the one loop minima of the effective potential are generically located at a nonperturbative mass scale, again invalidating the reliability of the perturbative analysis. Our results described above for negative values of the conformal mass term highlight the inadequacy of a perturbative treatment.

The layout of this paper is as follows. Section 2 introduces the O(N) model on a general background, including effects due to compactness of the spacetime. Section 3 specialises to the squashed three-sphere and studies the conformally coupled theory. Section 4 connects the field theory results with the dual gravitational description using the Klebanov-Polyakov correspondence. The appendices contain technical calculations of the zeta function on a squashed three-sphere, as well as a check on our calculations.

2. The O(N) model at large N

We begin by reviewing known results on the interacting O(N) scalar field theory on a D dimensional spacetime $\mathcal{M}$ (see e.g. [16]). We also note some subtleties arising from the fact that the spacetime $\mathcal{M}$ is compact. The large N limit of the field theory will allow us to analyse the exact effective action for this theory and thus draw reliable conclusions about the dynamics of the quantum theory. In Euclidean signature the O(N) model in D dimensions has the classical action

$$S_{cl}[\vec{\Phi}] = \int d^D x\sqrt{g}\left[\frac{1}{2}\nabla\vec{\Phi}\cdot\nabla\vec{\Phi} + \frac{1}{2}m^2\,\vec{\Phi}^2 + \frac{\lambda}{4N}\left(\vec{\Phi}\cdot\vec{\Phi}\right)^2\right], \tag{2.1}$$

where $\vec{\Phi}$ is an N-component field transforming as a vector under $\mathrm{O}(N)$ rotations. The interacting scalar field theory is super-renormalisable in dimensions less than four. The explicit N dependence in (2.1) is necessary for the theory to posses a well defined large N limit. In the action (2.1) the mass term includes a possible ξR coupling to the scalar curvature of the background.

To solve the large N theory one introduces an auxiliary scalar field σ which permits a rewriting of the action as

$$S_{cl}[\vec{\Phi}] = \int d^D x\sqrt{g}\left[\frac{1}{2}\nabla\vec{\Phi}\cdot\nabla\vec{\Phi} + \frac{1}{2}\left(m^2+\lambda\sigma\right)\vec{\Phi}\cdot\vec{\Phi} - \frac{\lambda N}{4}\sigma^2\right]. \tag{2.2}$$

The field σ is the composite operator,

$$N\sigma = \vec{\Phi}\cdot\vec{\Phi}\,. \tag{2.3}$$

To derive the large N effective potential we introduce a homogeneous background expectation value for the $\mathrm{O}(N)$ field and integrate out all inhomogeneous fluctuations about this configuration. Without loss of generality we may always choose to rotate this expectation value into the top component of the $\mathrm{O}(N)$ vector, so that

$$\vec{\Phi} = (\sqrt{N}\phi + \delta\phi,\ \pi_1,\ \pi_2, \ldots,\ \pi_{N-1})\,, \tag{2.4}$$

where ϕ is a homogeneous background and $\delta\phi$ and $\vec{\pi}$ are the quantum fluctuations. As we will see shortly, the explicit factor of $\sqrt{N}$ in (2.4) is the correct scaling behaviour for VEVs in the large N theory so that $\phi \sim \mathcal{O}(N^0)$. At this point it is worth noting that the backgrounds we consider in this paper, squashed three-spheres, are homogeneous and hence the notion of an effective potential for homogeneous fields makes sense.

Introducing a homogeneous background might appear to break the $\mathrm{O}(N)$ symmetry to $\mathrm{O}(N-1)$. However, the path integral must include an integral over the vacuum manifold and we will see that this implies that symmetry breaking does not actually occur on a compact space. In fixing the $\mathrm{O}(N)$ symmetry for the homogeneous modes in (2.4) we have effectively transformed into polar coordinates for $\vec{\Phi}$ with $\sqrt{N}\phi$ playing the role of the radial coordinate. Upon integrating over the angular directions in field space the partition function picks up an extra factor given by the volume of the vacuum manifold $\mathrm{O}(N)/\,\mathrm{O}(N-1) = S^{N-1}$

$$\frac{2\pi^{(N-1)/2}}{\Gamma[(N-1)/2]}\left(N^{1/2}\phi\right)^{N-1} \overset{N\to\infty}{=} \phi^N\pi^{N/2}e^{N/2} = \exp\left(\frac{N}{2}\left[1+\ln\pi+\ln\left(\frac{\phi^2}{\mu}\right)\right]\right), \tag{2.5}$$

where μ is an arbitrary scale introduced to keep the arguments of the logarithms dimensionless. We note that there is a curious cancellation between the N^N term coming from the radius of the N-sphere and from the large N limit of the Gamma function. The result of this cancellation is that the extra contribution to the partition function (2.5) has the same N dependence as the remaining terms in the effective action in the large N limit.

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In the large N limit the effective potential for the homogeneous fields ϕ and σ, after integrating out $\vec{\pi}$ and including the contribution (2.5), is[1]

$$\frac{V_{\text{eff}}(\phi,\sigma)}{N} = \frac{1}{2}\left(m^2+\lambda\sigma\right)\phi^2 - \frac{\lambda}{4}\sigma^2 + \frac{1}{2\text{Vol}\mathcal{M}}\ln\det{}'\left[\frac{-\Box+m^2+\lambda\sigma}{\mu^2}\right] - \\ - \frac{1}{2\text{Vol}\mathcal{M}}\left[1+\ln\pi+\ln\frac{\phi^2}{\mu}\right], \tag{2.6}$$

where Vol$\mathcal{M}$ is the volume of the spacetime and μ is again an arbitrary dimensionful scale which should be interpreted as a sliding renormalisation scale. The prime in det$'$ denotes the fact that the integration over the $\vec{\pi}$ fluctuations did not include the constant modes, which we have dealt with separately.

In the large N limit only the saddle point configuration obtained by extremising (2.6) contributes to the partition function. Extremising $V_{\text{eff}}(\phi,\sigma)$ with respect to the background fields ϕ and σ we then find the vacuum equations

$$\phi^2(m^2+\lambda\sigma) = \frac{1}{\text{Vol}\mathcal{M}}\,, \tag{2.7}$$

and

$$\left(\phi^2-\sigma\right) + \frac{1}{\text{Vol}\mathcal{M}}\,\text{Tr}'\left[\frac{1}{-\Box+m^2+\lambda\sigma}\right] = 0\,. \tag{2.8}$$

The second of these equations is in fact, in the large N limit, just the vacuum expectation value of the operator equation (2.3) which would yield $\sigma = \phi^2 + \langle\vec{\pi}^2\rangle/N$. Here $\vec{\pi}$ refers to the $(N-1)$ "pions" of equation (2.4). Thus the functional trace computes the renormalised vacuum expectation value of the composite operator $\vec{\pi}\cdot\vec{\pi}$, encoding the contribution of the quantum fluctuations about the ground state.

At this stage it is convenient to define the "effective pion mass"

$$m_\pi^2 = m^2 + \lambda\sigma\,, \tag{2.9}$$

which is precisely the mass of the $(N-1)$ "pion" fluctuations in (2.4). From equation (2.8) we see that this effective mass incorporates a self-consistent resummation of the quantum fluctuations about the vacuum state. This is a consequence of the large N limit which resums the so-called cactus diagrams of the theory.

The effective mass m_π^2 depends on whether the theory is realised in a symmetric phase or not. In an O(N)-symmetric phase m_π^2 is nonvanishing. In a symmetry broken phase we would expect m_π^2 to vanish, thus implementing Goldstone's theorem. That is, in a symmetry broken phase there would be $N-1$ massless Goldstone bosons. In equation (2.7) we see that m_π^2 cannot be zero. To achieve this we would need to take Vol$\mathcal{M}$ or ϕ^2 to infinity. We will see below that taking ϕ^2 to infinity requires taking the coupling to zero. Thus we find that symmetry breaking cannot occur for the O(N) model in a compact spacetime at large N and finite coupling.

[1]The contribution from integrating out the fluctuations $\delta\phi$ is subleading in the $1/N$ expansion.

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Equations (2.7) and (2.8) may be rewritten as a gap equation for m_π^2

$$\boxed{m_\pi^2 = m^2 + \frac{\lambda}{\mathrm{Vol}\mathcal{M}} \mathrm{Tr}\left[\frac{1}{-\Box + m_\pi^2}\right].} \tag{2.10}$$

Note that the trace now includes the constant mode, as will all remaining traces in this paper. Once we have solved (2.10) for m_π^2 we may evaluate the effective potential at the extremum to find the action

$$\begin{aligned} I &= \int d^D x \sqrt{g}\, V_{\text{eff.}} \\ &= \frac{N}{2}\left[-\frac{\mathrm{Vol}\mathcal{M}}{2\lambda}(m^2 - m_\pi^2)^2 + \ln\det\left(\frac{-\Box + m_\pi^2}{\mu^2}\right) + \ln(\mu^3 \mathrm{Vol}\mathcal{M})\right] + \text{const.} \end{aligned} \tag{2.11}$$

The only effect of scaling μ is to change the unphysical additive constant in the action.

2.1 Dynamics on a curved background $\mathcal{M}$

To compute the large N effective potential and its minima, we need to evaluate the functional traces of equations (2.10) and (2.11) on $\mathcal{M}$. A natural way of doing this is by using zeta function regularisation. Recall that the zeta function of an elliptic operator A is defined by

$$\zeta(s) = \mathrm{Tr}\, A^{-s}\,. \tag{2.12}$$

Thus for the operator $[-\Box + m_\pi^2]/\mu^2$ on $\mathcal{M}$ we set

$$\ln\det\left(\frac{-\Box + m_\pi^2}{\mu^2}\right) = -\lim_{s\to 0}\frac{d}{ds}\mathrm{Tr}\left(\frac{-\Box + m_\pi^2}{\mu^2}\right)^{-s} \equiv -\zeta'(0)\,. \tag{2.13}$$

Differentiating this expression one obtains

$$\mathrm{Tr}\left[\frac{1}{-\Box + m_\pi^2}\right] = \frac{1}{\mu^2}\lim_{s\to 0}\frac{d}{ds}(s\,\zeta(s+1))\,. \tag{2.14}$$

The main technical aspects of this work will involve the computation of the zeta function on a squashed three-sphere. The methods we use are similar to those in [23], although we are working in a nonperturbative framework.

3. O(N) model on a squashed three-sphere

In this section we apply the above formalism with a squashed three-sphere as the background. The metric of the squashed three-sphere of radius a is

$$ds^2 = \frac{a^2}{4}\left(\sigma_1^2 + \sigma_2^2 + \frac{1}{1+\alpha}\sigma_3^2\right), \tag{3.1}$$

where σ_i are the usual left-invariant $SU(2)$ one forms

$$\sigma_1 + i\sigma_2 = e^{-i\psi}\left(d\theta + i\sin\theta d\phi\right)\,, \quad \sigma_3 = d\psi + \cos\theta d\phi\,, \tag{3.2}$$

and α is the squashing parameter which may take values in the range $\alpha \in (-1, \infty)$. In this parameterisation $\alpha = 0$ is the round three-sphere. The other manifolds are also topologically spheres, but with a squashed S^1 fibration over S^2. The scalar curvature is found to be

$$R = \frac{2(3+4\alpha)}{a^2(1+\alpha)} \,. \tag{3.3}$$

Note that the scalar curvature changes sign and becomes negative when $\alpha < -3/4$. We will also need the volume

$$\mathrm{Vol}(\mathcal{M}) = \frac{2a^3\pi^2}{(1+\alpha)^{1/2}} \,. \tag{3.4}$$

Although squashing the round sphere introduces an anisotropy, it preserves the homogeneity of the space. Hence the spectrum of the scalar laplacian on (3.1) is straightforward to calculate. As we saw above, this is what we need in order to compute the quantum correction to the self-energy which then enters the right hand side of the gap equation (2.10) of the large N theory on the squashed sphere. Following some minor manipulations [23], the zeta function (2.12) for the operator $(-\Box + m_\pi^2)/\mu^2$ on the squashed sphere may be written as

$$\zeta(s) = \sum_{l=1}^{\infty}\sum_{q=0}^{l-1} \frac{l(a\mu)^{2s}}{[l^2 + \alpha(l-1-2q)^2 + a^2m_\pi^2 - 1]^s} \,. \tag{3.5}$$

We may obtain a finite expression for $\zeta(s)$ by analytic continuation of the sum (3.5). The methods are standard and are described in appendix A. We find

$$\begin{aligned} \frac{1}{\mu^{2s}}\zeta(s) &= \frac{1}{m_\pi^{2s}} + \frac{4a^{2s}}{(3+a^2m_\pi^2+\alpha)^s} + a^{2s}\Theta(\alpha)H + \\ &\quad + a^{2s}\int_0^1 \frac{F(y)dy}{(1+\alpha y^2)^s} + \frac{a^{2s}}{(1+\alpha)^s}\int_0^\infty \frac{G(y)dy}{e^{2\pi y}+1} \,, \end{aligned} \tag{3.6}$$

where the functions F, G and H are given in appendix A and $\Theta(\alpha)$ is the Heaviside step function. We point out that although the appearance of the step function might suggest a discontinuity in the zeta function at $\alpha = 0$, the function H and all its derivatives actually vanish exponentially at $\alpha = 0$, see equation (A.23) below. In fact the connection between the squashed sphere and thermal field theory is seen explicitly in the function H and $dH(s)/ds|_{s=0}$ (see appendix A). The latter has a natural interpretation as the finite temperature free energy of a field theory on S^2 accompanied by certain parity reversals.

For technical reasons in performing the analytic continuation, again see appendix A, we have restricted ourselves to $-8/9 < \alpha < \infty$, omitting the range $-1 < \alpha \leq -8/9$. The range we consider includes the value $\alpha = -3/4$, where the Ricci scalar changes sign, the value corresponding to the round sphere, $\alpha = 0$, and also arbitrarily large values of α that are relevant for the Hawking-Page type transition in the dual gravitational theory. From our results, the extrapolation to the range $-1 \leq \alpha < -8/9$ seems straightforward, although the $\alpha \to -1$ limit itself is fairly singular and may contain interesting physics that deserves to be studied separately.

3.1 Physics of the symmetric vacuum I: the mass gap

In three dimensions, the coupling constant λ has mass dimension one and thus the interaction is a relevant operator which generates a nontrivial renormalisation group flow from a free fixed point in the UV. High energies correspond to weak coupling, and at long wavelengths the coupling becomes stronger. On the squashed sphere with scale size a a natural dimensionless coupling $a\lambda$ can be defined. We investigate the theory at different values of this dimensionless coupling, including arbitrarily strong and weak coupling regimes. Varying this coupling naturally corresponds to varying the size of the compact space keeping λ fixed. For any value of the coupling, the IR physics of the interacting large N theory is encapsulated in the solution to the mass gap equation (2.10).

In three dimensions, zeta functions are regular at $s = 1$. Therefore the right hand side of expression (2.14) reduces to $\zeta(1)$. The gap equation (2.10) then becomes

$$m_\pi^2 = m^2 + \frac{\lambda}{\mu^2 \mathrm{Vol}\mathcal{M}} \zeta_{m_\pi^2}(1)\,. \tag{3.7}$$

To ensure that in the ultraviolet our theory flows to a free conformal fixed point, we set the bare mass term, m^2, to be given by the conformal coupling to the background curvature R

$$m^2 = m^2_{\text{conf.}} = \frac{1}{8}R = \frac{3+4\alpha}{4a^2(1+\alpha)}\,. \tag{3.8}$$

The gap equation may now be written as

$$a^2 m_\pi^2 = \frac{3+4\alpha}{4(1+\alpha)} + \frac{a\lambda(1+\alpha)^{1/2}}{2\pi^2}\frac{\zeta_{m_\pi^2}(1)}{(a\mu)^2}\,. \tag{3.9}$$

The only dependence on a, the size of the squashed sphere, in this expression is contained in the dimensionless masses and couplings $a\lambda$ and $a^2 m_\pi^2$.

It is clear that (3.7) always has solutions with $m_\pi^2 > 0$, essentially due to the $1/m_\pi^2$ behaviour in $\zeta_{m_\pi^2}(1)$ as $m_\pi^2 \to 0$. In figure 1 we plot the solution for $a^2 m_\pi^2$ as a function of α for three fixed values of $a\lambda$. The existence of a symmetric vacuum for all couplings $a\lambda$ could only be seen because we have solved the theory nonperturbatively. At a perturbative level there is no symmetric phase for $\alpha < -3/4$ and small $a\lambda$.

An interesting feature of figure 1 is that the value of $a^2 m_\pi^2$ on the round sphere, $\alpha = 0$, is precisely the conformal value for $\alpha = 0$, $a^2 m_\pi^2 = 3/4$, for all values of the coupling. This can be checked analytically using the expressions for the zeta functions in appendix A. Thus the theory with tree level conformal mass term $a^2 m^2 = 3/4$ posesses a self-consistent solution to the gap equation, for any non-zero coupling $a\lambda$, precisely at $a^2 m_\pi^2 = 3/4$, which is again the conformal value! This can be understood as follows. On $\mathbb{R}^3$ the critical O(N) model at large N is obtained when m_π^2 vanishes. The round S^3 is conformally equivalent to $\mathbb{R}^3$. Thus it is natural that m_π^2 should take the conformal value on the round sphere. In fact this provides a check of our analytically continued expressions and will have a nice dual gravitational interpretation below.

We now turn to the strong and weak coupling results for the theory which may also be interpreted as large and small volume regimes.

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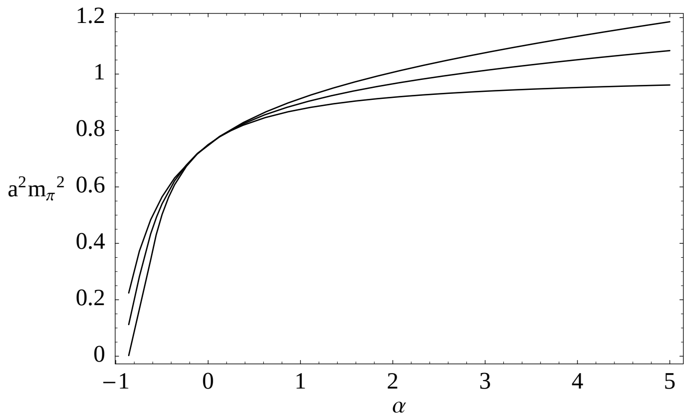

Figure 1: $a^2m_\pi^2$ as a function of the squashing α with, from bottom to top on the left of the graph, $a\lambda = 0.1$, 10 and 1000.

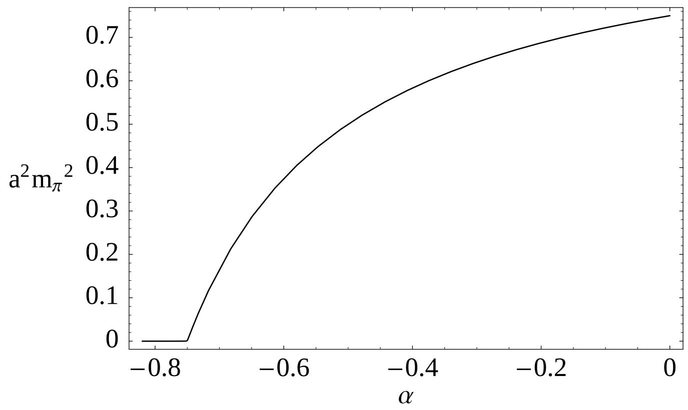

Figure 2: $a^2m_\pi^2$ as a function of the squashing $\alpha < 0$ and with $a\lambda \to 0^+$.

3.1.1 Weak coupling

In the weak coupling regime we may solve for m_π^2 analytically. From the fact that $\zeta(1) \sim 1/m_\pi^2$ as $m_\pi^2 \to 0$ and from the gap equation (3.7) it follows that

$$\begin{aligned} m_\pi^2 &\approx \frac{-\lambda}{m^2 \mathrm{Vol}\mathcal{M}} \quad \text{for} \quad a\lambda \ll 1 \quad \text{if} \quad \alpha < \frac{-3}{4}\,, \\ m_\pi^2 &\approx m^2 + \mathcal{O}(\lambda)\,, \quad \text{for} \quad a\lambda \ll 1 \quad \text{if} \quad \alpha > \frac{-3}{4}\,. \end{aligned} \tag{3.10}$$

We can see that as $a\lambda \to 0^+$ the derivative of m_π^2 becomes discontinuous at $\alpha = -3/4$. This is illustrated in figure 2.

The discontinuity suggests a phase transition at $\alpha = -3/4$ in the extreme weak coupling limit $a\lambda \to 0^+$. Indeed $m_\pi^2 \to 0$ in this limit. The vanishing of the "pion" mass, m_π^2, is reminiscent of symmetry breaking leading to massless Goldstone bosons. However, on a compact space there will be no symmetry breaking. Nevertheless, in the strict weak coupling limit there is a new phase that appears for $\alpha < -3/4$. We know from (2.7) that at finite volume this must require $\phi^2 \to \infty$. Indeed we will see below that this phase transition is characterized by an order parameter, namely the condensate $\lambda < \vec{\Phi}\cdot\vec{\Phi} >$ which vanishes for $\alpha < -3/4$ and is nonvanishing and of order N for $\alpha > -3/4$. However, in most of this paper we will be interested in the case of finite coupling in which this phase transition is completely smoothed out. The nonperturbative smoothing of phase transitions may be a more general phenomenon of relevance in attempts to match perturbative phase transitions with strong coupling results [13, 14].

3.1.2 Strong coupling

For any finite value of the coupling the gap equation (2.10) seems difficult to solve analytically. In the limit of infinite coupling, the gap equation simply becomes

$$\zeta_{m_\pi^2}(1) = 0 \qquad \text{as} \qquad a\lambda \to \infty\,. \tag{3.11}$$

Again, this equation does not appear easy to solve for general α but the roots can be determined numerically. However, it turns out to be analytically tractable at large α leading to a rather suggestive solution. Using the large α expression for the zeta function discussed in appendix C we find that m_π^2 grows linearly with α

$$\boxed{a^2 m_\pi^2 \approx \frac{\alpha}{\pi^2}\ln^2\left[\frac{1+\sqrt{5}}{2}\right] \qquad \text{as} \qquad \alpha \to \infty\,.} \tag{3.12}$$

The appearance of the golden mean in this expression is somewhat curious. Interpreting $\sqrt{\alpha}$ as a temperature T, this equation gives us the high temperature behaviour of the solution to the gap equation, $m_\pi^2 \sim T^2$. In fact, following the technique outlined in appendix C one obtains exactly the same expression as above for the $\mathrm{O}(N)$ model on $S^1 \times S^2$ in the limit of high temperature, or shrinking S^1. In this $S^1 \times \mathbb{R}^2$ limit, the expression (3.12) for the mass in terms of the golden mean was found previously in the $\mathrm{O}(N)$ model in [24].

3.2 Physics of the symmetric vacuum II: the free energy

Using the solution to the large N gap equation we compute the value of the action of the model as a function of α. We may do this keeping the volume of the squashed S^3 fixed while varying the squashing parameter. This has the effect of removing the logarithmic dependence of the action (2.11) on the volume of the squashed sphere. This action has a natural thermodynamical interpretation as the free energy times the inverse temperature. In figure 3 we plot the action of the symmetric vacuum (2.11) against α for strong coupling $a\lambda \to \infty$. The qualitative form of the plot remains unchanged at finite values of the coupling, although it does change for weak coupling.

The interesting features of this plot include the behaviour at large positive α, the behaviour near $\alpha = -1$ and the presence of a maximum at $\alpha = 0$. Based on numerical analysis, we are able to deduce that near $\alpha = -1$, the action goes like $I \sim -1/(1+\alpha)^2$. For large positive α the action scales linearly with α, which may be shown analytically. We will see below that all three features are also encountered in the dual gravitational theory.

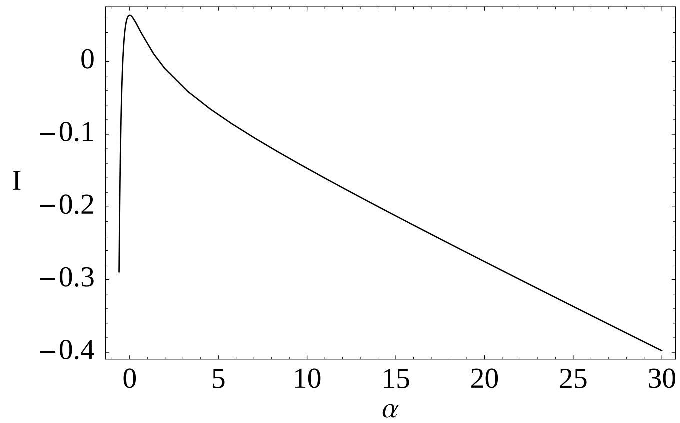

Figure 3: The field theory action I as a function of the squashing α with $a\lambda \to \infty$.

We may find a closed expression for the free energy at large squashing α using (3.12) for strong coupling and (3.10) for weak coupling. The techniques discussed in appendix C yield

$$\boxed{I\,\big|_{a\lambda\ll 1} \approx -\frac{N\zeta_R(3)}{8\pi^2}\,\alpha \qquad \text{as} \qquad \alpha\to\infty\,,} \tag{3.13}$$

and also remarkably

$$\boxed{I\,\big|_{a\lambda\to\infty} \approx \frac{4}{5}\,I\,\big|_{a\lambda\ll 1} \qquad \text{as} \qquad \alpha\to\infty\,.} \tag{3.14}$$

In these expressions $\zeta_R(s)$ denotes the Riemann zeta function. If one considers the squashed S^1 direction as a nontrivially fibred temperature, then the linear dependence on α is interpreted as a T^2 scaling with temperature. Writing $F = IT$, this gives the usual high temperature scaling of the free energy in a three dimensional spacetime.

The factor of 4/5 difference between the strong and weak coupling regimes is reminiscent of the well known 3/4 factor that distinguishes the strong and weak coupling limits of the free energy of $\mathcal{N} = 4$ super Yang Mills theory in four dimensions [18]. In the present context we are able to calculate both limits nonperturbatively within field theory. The presence of $\zeta_R(3)$ in the free energies (3.13) and (3.14) is generic for conformal field theories on $S^1 \times \mathbb{R}^2$, the high temperature limit, see [24] and references therein. That paper also finds a factor of 4/5 in the free energy of the O(N) model on $S^1 \times \mathbb{R}^2$.

4. The gravitational dual

According to the original correspondence proposed by Klebanov and Polyakov [4] the gravitational theory dual to the three dimensional critical O(N) vector model is a higher spin gauge theory in AdS$_4$ spacetime [25 − 27]. The details of this bulk theory, and hence the correspondence itself, have not yet been completely understood for AdS$_4$ spacetime whose boundary in global coordinates is the round three-sphere [28]. For the O(N) model on squashed spheres, the dual gravitational description should be a higher spin gauge theory on bulk geometries whose conformal boundaries are squashed three-spheres. We leave an in-depth study of the higher spin theory to future work. Instead, we will test the exact results from our field theory analysis against known semiclassical results regarding the associated bulk geometries, without taking into account the effect of any higher spin degrees of freedom. We find a remarkably detailed qualitative agreement between the two pictures.

In general there are two four dimensional riemannian geometries with negative cosmological constant that have the squashed three-sphere as conformal boundary. These are the AdS Taub-NUT and the AdS Taub-Bolt spacetimes. In Einstein semiclassical quantum gravity there is a first order phase transition between the two geometries at a critical value of the squashing parameter [5, 6]. This phase transition is simply the NUT charged version of the Hawking-Page transition for black holes in AdS [12]. More concretely, one finds that the AdS Taub-Bolt geometry only exists for $\alpha \geq 5 + 3\sqrt{3} \approx 10.2$ and the quantum gravity phase transition itself occurs at $\alpha_{\text{crit}} = 6 + 2\sqrt{10} \approx 12.3$ [5, 6]. The authors of [5, 6] computed the difference of the actions associated to the AdS Taub-NUT and AdS Taub-Bolt spacetimes and inferred the existence of a transition at the point where the difference vanishes. More interestingly from our point of view, in [7] the individual actions for the AdS Taub-NUT and AdS Taub-Bolt geometries were obtained using a boundary countertem technique inspired by the prescription of [29]. We quote the result of [7] for the bulk action as a function of the boundary squashing parameter

$$I_{\text{TN}} = -\frac{6\pi}{GR}\frac{(2\alpha+1)}{(\alpha+1)^2}\,, \tag{4.1}$$

where G is Newton's constant and R is the (negative) Ricci scalar for AdS Taub-NUT. Note that we use R to denote both the boundary and the bulk scalar curvatures. This solution exists for the entire range of α and dominates the partition function for $\alpha < 6 + 2\sqrt{10}$. In particular near $\alpha = -1$ the bulk action behaves as $I_{\text{TN}} \sim -1/(1+\alpha)^2$ which is precisely what we found in the strongly coupled field theory. In addition I_{TN} has a maximum at $\alpha = 0$, mirroring our field theory result of figure 3.

For $\alpha > 6 + 2\sqrt{10}$ the bulk action is that of AdS Taub-Bolt

$$I_{\text{TB}} = -\frac{24\pi}{RG}\,(1+\alpha)^{-1/2}\,\left(m_b + \frac{3}{4}\,r\,(1+\alpha)^{-1} - r^3\right), \tag{4.2}$$

with

$$m_b = \frac{1}{2}r + \frac{1}{8r}(1+\alpha)^{-1} + \frac{1}{2}\left(r^3 - \frac{3}{2}r(1+\alpha)^{-1} - \frac{3}{16r}(1+\alpha)^{-2}\right), \tag{4.3}$$

and

$$r = \frac{1}{6}(1+\alpha)^{1/2}\left(1 + \sqrt{1 - 12(1+\alpha)^{-1} + 9(1+\alpha)^{-2}}\right). \tag{4.4}$$

For large α, the Bolt action is negative and grows linearly

$$I_{\text{TB}} \approx \frac{4\pi}{9GR}\alpha \qquad \text{as} \qquad \alpha \to \infty\,. \tag{4.5}$$

This linear behaviour of the action translates into a T^2 scaling law with temperature, and hence a T^3 scaling of the free energy with temperature. This is essential in order to have an interpretation in terms of a three dimensional boundary field theory. Indeed,

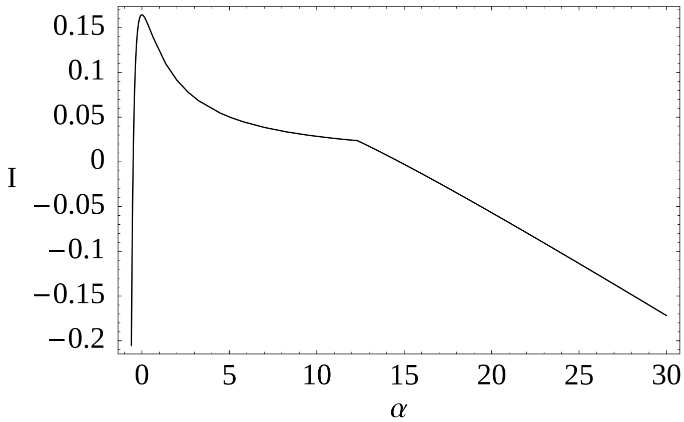

Figure 4: The bulk action I as a function of the squashing α.

our field theory result (3.14) exhibits a negatively sloped linear dependence on α for large squashings. On the other hand, the AdS Taub-NUT solution vanishes for large α as $I_{\mathrm{TN}} \sim 1/\alpha$.

In figure 4 we have plotted the bulk action as a function of α. We have normalised the action so that its slope at large α agrees with the field theory result (3.14). There is an additive constant in the action which could be adjusted to make the peak values of the actions match. Comparing with figure 3, which was obtained from the strongly coupled field theory, we clearly see the qualitative similarities at small and large values of α. However, it is clear that there is no field theory analog of the Hawking-Page transition seen in the bulk. As we have pointed out earlier, we can presumably attribute this to the fact that the infinite massless higher spin degrees of freedom have not been taken into account in the bulk analysis.

We should emphasise that the qualitative behaviour of the field theory at very weak coupling is quite different to the strong coupling regime, particularly for the negative values of α where the boundary curvature becomes negative. At weak coupling we find that the action drops rapidly for $\alpha < -3/4$ and scales as $-1/(1+\alpha)^{5/2}$ which is faster than the gravity result which scales like $-1/(1+\alpha)^2$ near $\alpha = -1$. This is fairly straightforward to see following our treatment of the weak coupling regime. The strongly coupled field theory on the other hand, based on numerical evidence, appears to match the behaviour of the bulk theory near $\alpha = -1$. This suggests, as is usual in AdS/CFT dualities, that the strongly coupled field theory is dual to gravity on a weakly curved space. Below we will present another piece of evidence in favour of this identification which pertains to the behaviour of the field theory condensate $\langle \vec{\Phi} \cdot \vec{\Phi} \rangle$ in the strongly coupled field theory and its implications for the gravity dual.

Equating the bulk and boundary values for the effective action at large α, (3.14) and (4.5), suggests the tentative dictionary

$$- RG = \frac{\pi^3 40}{9\zeta_R(3)} \frac{1}{N} \,. \tag{4.6}$$

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This formula passes the immediate test of giving weak curvatures when N is large. It might be compared with the well known result $RG \sim 1/N^{3/2}$ for the $AdS_4 \times S^7$ version of the correspondence. The appearance of $\zeta_R(3)$ in (4.6) is curious and possibly tantalising given that $\zeta_R(3)$ also appears in a computation of α' corrections to the IIB effective supergravity action and hence the bulk free energy [30] for the $AdS_5 \times S^5$ version of the correspondence. In general we expect the coefficient in (4.6) to depend on the background. The same $RG \sim 1/N$ relation was found for the Klebanov-Polyakov correspondence on AdS_4 [28].

4.1 Negative curvature of the boundary

Let us consider the physics associated with the sign change in the scalar curvature of the boundary at $\alpha = -3/4$. From our comments above, the dual geometry to the squashed three-sphere in this regime is the AdS Taub-NUT spacetime which may be written as

$$ds^2 = \frac{1}{k^2(1-r^2)^2}\left[\frac{4(1+\alpha r^2)}{1+\alpha r^4}dr^2 + r^2(1+\alpha r^2)(\sigma_1^2+\sigma_2^2) + r^2\frac{1+\alpha r^4}{1+\alpha r^2}\sigma_3^2\right], \qquad (4.7)$$

where the cosmological constant is $\Lambda = -3k^2$ and the range of the radial coordinate is $0 \leq r < 1$. We have already argued above that the weakly curved gravitational description appears to be dual to the strongly coupled $\mathrm{O}(N)$ model. Therefore we expect that the fact that a stable symmetric phase continues to exist for $\alpha < -3/4$ at strong coupling should translate into the bulk spacetime remaining stable in this regime. We have also seen that there is a phase transition at $\alpha = -3/4$ for arbitrarily weak coupling. However, we expect this to be a strong curvature effect in the gravitational dual and thus beyond a classical gravitational computation.

In string theory duals to field theories there are generically two types of instabilities that can arise when the conformal boundary has negative scalar curvature. For a summary of these, see [10]. The first is a nonperturbative instability due to the nucleation of BPS branes. The second is a perturbative instability due to negative modes of fluctuations about the solution. It is unclear whether the bulk higher spin gauge theory will contain objects analogous to BPS branes, so we will focus on the perturbative possibility.

The perturbative stability of the AdS Taub-NUT spacetime against fluctuations of a scalar field was considered in [10]. That paper was primarily interested in scalar fields with a mass saturating the Breitenlohner-Freedman bound, as these arise from the compactification of eleven dimensional supergravity to four dimensions. The higher spin gauge theory that is conjectured to be dual to the $\mathrm{O}(N)$ model has instead a conformally coupled scalar field, φ. It is simple to adapt the analysis of [10] to this case. We are interested in whether the bulk equation

$$-\Box\varphi + \frac{R}{6}\varphi = \beta\varphi\,, \qquad (4.8)$$

has solutions with negative β. This equation may be separated and converted into Schrödinger form

$$-\frac{d^2\chi}{dr_*^2} + V(r_*)\chi = \beta\chi\,, \qquad (4.9)$$

using the transformations

$$\chi = (gg^{rr})^{1/4}\varphi\,, \qquad dr = (g^{rr})^{1/2}dr_*\,, \tag{4.10}$$

where g is the determinant of the metric (4.7). It is then easy to see that the potential $V(r_*)$, is everywhere positive for all values of the squashing α. Therefore β cannot be negative and the spacetime is perturbatively stable.

One should also check stability against the other fields in the theory, such as metric fluctuations. However, stability of the bulk would be consistent with our field theory finding that the symmetric phase remains stable at finite coupling as we go to the regime of parameter space with a negatively curved boundary.

4.2 The condensate $\langle\vec{\Phi}\cdot\vec{\Phi}\rangle$

Finally, we turn to an analysis of the condensate $\langle\vec{\Phi}\cdot\vec{\Phi}\rangle$ in field theory and its implications for bulk physics. We will also argue that the condensate can be used to construct an order parameter for the extreme weak coupling phase transition discussed in section 3.1.1.

From the defining equations for the composite operator σ and pion mass m_π^2, (2.3) and (2.9), we have that

$$\frac{1}{N}\langle\vec{\Phi}\cdot\vec{\Phi}\rangle = \frac{1}{\lambda}\left(m_\pi^2 - m^2\right)\,. \tag{4.11}$$

The Klebanov-Polyakov duality [4] then relates the vacuum expectation value $\langle\vec{\Phi}\cdot\vec{\Phi}\rangle$ to the normalisable mode of the conformally coupled bulk scalar field φ.

Let us firstly consider the strong coupling limit. We saw in equation (3.11) that $m_\pi^2 - m^2 = \mathcal{O}(1)$ as $a\lambda \to \infty$, that is, m_π^2 remains finite as we take the strong coupling limit. Thus we have

$$\frac{1}{N}\langle\vec{\Phi}\cdot\vec{\Phi}\rangle \;\sim\; \mathcal{O}\left(\frac{1}{\lambda}\right) \qquad \text{as} \qquad a\lambda \to \infty\,. \tag{4.12}$$

This implies that at strong coupling, $a\lambda \to \infty$, the condensate vanishes. If we identify the strong coupling regime of the O(N) model with the weakly curved gravitational dual, then this result translates into the (true) statement that the bulk field φ is not turned on in the AdS Taub-NUT and AdS Taub-Bolt geometries.

The nonvanishing of the condensate at finite coupling then translates into the prediction that the bulk must allow for a nonvanishing value for φ in these cases. It is likely that these will just be new backgrounds with a nonvanishing φ profile turned on self-consistently due to higher curvature corrections. Figure 5 shows a plot of $\langle\vec{\Phi}\cdot\vec{\Phi}\rangle$ as a function of the squashing at two values of the coupling.

One remarkable aspect of figure 5, which was anticipated already in figure 1, is that the condensate vanishes for the round sphere $\alpha = 0$ for all values of the coupling. The dual implication of this fact is that when the bulk geometry is simply AdS, as opposed to AdS Taub-NUT or AdS Taub-Bolt, then it does not receive corrections at finite curvatures which turn on the scalar φ. Such protection against corrections seems plausible, given the maximal symmetry of AdS, and mirrors known results for other realisations of the AdS/CFT correspondence, see for instance [31] for the $AdS_5 \times S^5$ version of the correspondence.

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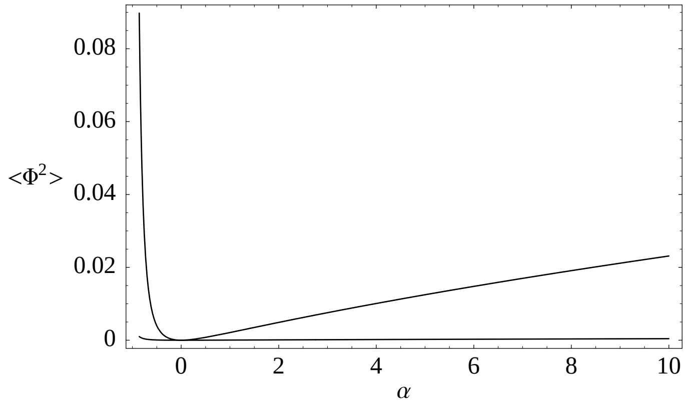

Figure 5: The condensate $\langle\vec{\Phi}\cdot\vec{\Phi}\rangle/N$ as a function of the squashing α with $a\lambda = 10$ (top) and $a\lambda = 1000$ (bottom).

At weak coupling the behaviour of the condensate (4.11) follows immediately from our previous results (3.10). In particular these imply that we may use the condensate as an order parameter for the symmetry breaking phase transition occuring at $\alpha = -3/4$ as $a\lambda \to 0^+$

$$\lim_{a\lambda\to 0^+}\left[\frac{\lambda}{N}\langle\vec{\Phi}\cdot\vec{\Phi}\rangle\right] = \begin{cases}\mathcal{O}(1) & \text{for} \quad \alpha < -3/4\,,\\ 0 \quad \text{for} & \alpha > -3/4\,.\end{cases} \tag{4.13}$$

5. Some open questions

Motivated by the duality conjectured by Klebanov and Polyakov [4], in this paper we have investigated and solved various aspects of the large N limit of the O(N) vector model on a squashed three-sphere. Perhaps surprisingly we have found nontrivial agreement between our field theory results and the semiclassical gravitational physics of the proposed dual geometries: AdS Taub-NUT and AdS Taub-Bolt.

There are various open ends to this work. The most obvious open question is the fate of the Hawking-Page transition in the bulk theory when massless higher spin degrees of freedom are taken into account. The field theory results suggest a smoothing out of the transition.

Within field theory, one limit which we have not considered in detail is the $\alpha \to -1$ limit. This is a rather singular limit, in which one dimension of the squashed sphere decompactifies. Nonetheless, perhaps there is interesting physics to be discovered. Our results near this limit have been mostly numerical. We have observed numerically that the strongly coupled field theory action in this regime scales in the same way as the action for the AdS Taub-NUT solution, with $I \sim -1/(1+\alpha)^2$. An analytical solution of the field theory problem in this regime may shed further light upon the agreement between field theory and gravity.

Finally, it would be interesting to know if there are deeper reasons underlying certain amusing numerical coincidences we observed in the exact solution to the strongly coupled theory. Namely the appearance of the golden mean in the solution (3.12) to the gap

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equation at high temperature, or large squashing, and the appearance of 4/5 as the ratio of the weak and strong coupling limits of the free energy (3.14) at large squashing. These results are essentially due to the O(N) model on $S^1 \times \mathbb{R}^2$ [24].

Acknowledgments

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A. Analytic continuation of the zeta function

The zeta function we wish to analytically continue is

$$\frac{1}{\mu^{2s}}\zeta(s) = \sum_{l=1}^{\infty}\sum_{q=0}^{l-1}\frac{la^{2s}}{[l^2+\alpha(l-1-2q)^2+a^2m_\pi^2-1]^s} \equiv \sum_{l=1}^{\infty}\sum_{q=0}^{l-1} f(q,l)\,. \tag{A.1}$$

The method used has two steps, the first turns the q sum into an integral and the second deals with the l summation [23].

The Abel-Plana formula converts the q sum into integrals. To apply this formula we need to know the location of the branch points of $f(q,l)$. These are

$$q_\pm = \frac{l-1}{2} \pm \frac{i}{2}\left(\frac{l^2-1+a^2m_\pi^2}{\alpha}\right)^{1/2}\,. \tag{A.2}$$

We see that the sign of α is important for determining the position of the branch points. Let us firstly consider the case $\alpha < 0$. In this case, the branch points are on the real axis and the Abel-Plana formula is

$$\sum_{q=0}^{l-1} f(q,l) = \int_{-\frac{1}{2}}^{l-\frac{1}{2}} f(t,l)dt - 2i\int_0^\infty \frac{dt}{e^{2\pi t}+1}\left[f\left(\frac{1}{2}+it,l\right) - f\left(\frac{1}{2}-it,l\right)\right], \tag{A.3}$$

so long as the branch points $q_\pm$ are outside the range of integration $(-1/2, l-1/2)$. For a given value of α this may be achieved by curtailing the range of l. We will restrict attention to $-8/9 < \alpha$ which will allow us to use the previous expression (A.3) for all values $l \geq 3$. The remaining terms in the sum corresponding to $l = 1, 2$ are then added to the final expression to obtain

$$\frac{1}{\mu^{2s}}\zeta(s) = \frac{1}{m_\pi^{2s}} + \frac{4a^{2s}}{(3+a^2m_\pi^2+\alpha)^s} + a^{2s}\int_0^1 \frac{F(y)dy}{(1+\alpha y^2)^s} + \frac{a^{2s}}{(1+\alpha)^s}\int_0^\infty \frac{G(y)dy}{e^{2\pi y}+1}, \tag{A.4}$$

with

$$F(y) = \sum_{l=3}^{\infty}\frac{l^2}{(l^2-A^2)^s}, \tag{A.5}$$

$$G(y) = 2i\sum_{l=3}^{\infty}\left[\frac{l}{([l+iB]^2-E^2)^s} - \frac{l}{([l-iB]^2-E^2)^s}\right], \tag{A.6}$$

where A, B, E are given by

$$
\begin{aligned}
A^2 &= \frac{1 - a^2 m_\pi^2}{1 + \alpha y^2}\,, \\
B &= \frac{2\alpha y}{1 + \alpha}\,, \\
E^2 &= \frac{1 - a^2 m_\pi^2}{1 + \alpha} + \frac{4\alpha y^2}{(1 + \alpha)^2}\,.
\end{aligned}
\tag{A.7}
$$

Note that E^2 changes sign at an intermediate value of y.

We now continue the sums in (A.5) using the Sommerfeld-Watson transformation. For the case of $F(y)$ we have

$$
F(y) = \frac{i}{2} \int_{C_1} dz \frac{z^2 \cot \pi z}{(z^2 - A^2)^s}\,,
\tag{A.8}
$$

where the contour C_1 is shown in figure 6. By considering sufficiently large s we may rotate this contour into C_2, also shown in figure 6, to obtain

$$
F(y) = \frac{i}{2} \int_{C_2} dz \frac{z^2 \cot \pi z}{(z^2 - A^2)^s}\,.
\tag{A.9}
$$

This contour integral will require one more analytic continuation along the imaginary axis using

$$
\coth \pi t = 1 + \frac{2}{e^{2\pi t} - 1}\,.
\tag{A.10}
$$

We are interested in $F(y)$ at $s = 0, 1$ and in its derivative with respect to s at $s = 0$. One finds, after paying due attention to the branch cuts, the values

$$
F(y)|_{s=1} = \frac{-\pi A}{2} \cot \pi A + \sum_{p=1}^{2} \frac{p^2}{A^2 - p^2}\,,
\tag{A.11}
$$

$$
F(y)|_{s=0} = -5\,.
\tag{A.12}
$$

The derivative at $s = 0$ may be shown to be

$$
\begin{aligned}
\left.\frac{d}{ds} F(y)\right|_{s=0} = &- \mathrm{Re}\Big[\frac{i\pi A^3}{3} + A^2 \ln\left(1 - e^{-2i\pi A}\right) + \\
&+ \frac{iA}{\pi} \mathrm{Li}_2\left(e^{-2i\pi A}\right) + \frac{1}{2\pi^2} \mathrm{Li}_3\left(e^{-2i\pi A}\right)\Big] + \sum_{p=1}^{2} p^2 \ln |A^2 - p^2|
\end{aligned}
\tag{A.13}
$$

In this expression Li_2 and Li_3 denote the dilogarithm and trilogarithm functions, respectively. It is sometimes simpler to avoid the polylogarithm functions. An alternative expression for when A is real is

$$
\left.\frac{d}{ds} F(y)\right|_{s=0} = -\pi P \int_0^A t^2 \cot \pi t dt - \frac{\zeta_R(3)}{2\pi^2} + \sum_{p=1}^{2} p^2 \ln |A^2 - p^2|\,,
\tag{A.14}
$$

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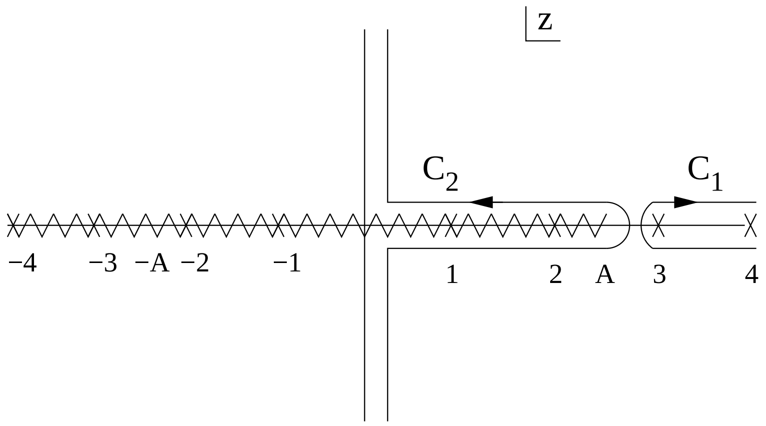

Figure 6: Change of contours for the analytic continuation.

whilst if A is imaginary we can write

$$\frac{d}{ds}F(y)\bigg|_{s=0} = \int_0^{|A|} \frac{2\pi t^2}{e^{2\pi t}-1}dt + \frac{i\pi A^3}{3} - \frac{\zeta_R(3)}{2\pi^2} + \sum_{p=1}^{2} p^2 \ln|A^2 - p^2| \,. \tag{A.15}$$

In these expressions $\zeta_R(s)$ is the Riemann zeta function and P denotes the Cauchy principal value of the integral.

Exactly the same manipulations are then performed for $G(y)$. The contours C_1 and C_2 are qualitatively as before, except that now the branch cuts emanate from the points $z = \pm E \pm iB$. The values are now

$$G(y)|_{s=1} = \frac{2\pi}{E}\frac{E\sinh(2\pi B) - B\sin(2\pi E)}{\cosh(2\pi B) - \cos(2\pi E)} - \sum_{p=1}^{2} \frac{8Bp^2}{|E^2 + B^2 - p^2 + i2pB|^2}\,, \tag{A.16}$$

$$G(y)|_{s=0} = 0\,. \tag{A.17}$$

The derivative is given by

$$\begin{aligned}\frac{d}{ds}G(y)\bigg|_{s=0} &= \sum_{p=1}^{2} i2p \ln\frac{(p+iB)^2 - E^2}{(p-iB)^2 - E^2} - 4\pi \int_0^{\infty} \frac{B\cosh 2\pi B - t\sinh 2\pi B - Be^{-2\pi t}}{\cosh 2\pi B - \cosh 2\pi t}dt + \\ &\quad + \sum_{\pm} \pm\Big[-2iE\ln\left(1 - e^{2\pi[\pm B + iE]}\right) + \frac{1}{\pi}\mathrm{Li}_2\left(1 - e^{\pm 2\pi B}\right) - \\ &\quad - \frac{1}{\pi}\mathrm{Li}_2\left(1 - e^{2\pi[\pm B + iE]}\right)\Big] + 2B\ln\frac{\cosh 2\pi B - 1}{\cosh 2\pi B - \cos 2\pi E}\,. \end{aligned} \tag{A.18}$$

In this last expression, $\mathrm{Li}_2(z)$ is the dilogarithm function. In evaluating these logarithms and dilogarithms, one should be careful to keep track of the phases of the complex numbers as E increases. In fact, the last two lines of (A.18) are given by

$$-4\pi \int_0^{E} \frac{B\sin 2\pi t - t\sinh 2\pi B}{\cosh 2\pi B - \cos 2\pi t}dt\,, \tag{A.19}$$

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when E is real and by

$$-4\pi|E|B + 4\pi \int_0^{|E|} \frac{B\cosh 2\pi B - t\sinh 2\pi B - Be^{-2\pi t}}{\cosh 2\pi B - \cosh 2\pi t} dt\,, \qquad \text{(A.20)}$$

when E is imaginary. These integral expressions are easier to evaluate numerically than the dilogarithms.

When $\alpha > 0$ the branch points $q_\pm$ are not on the real axis and $\mathrm{Re}\, q_\pm = (l-1)/2$ falls within the range of integration in (A.3). Application of the Abel-Plana formula is as for $\alpha < 0$ except that there is an extra contribution to the zeta function (A.4) given by an integral along the branch cuts, which we take to run from $q_\pm$ to $\pm i\infty$, respectively [23]. The extra term is found to be

$$a^{2s}H = -\frac{4a^{2s}}{(4\alpha)^s}\sin\pi s \sum_{l=3}^{\infty}\int_P^\infty \frac{l}{[1+(-1)^l e^{2\pi y}]}\frac{dy}{(y^2-P^2)^s}\,, \qquad \text{(A.21)}$$

where

$$P = \frac{1}{2}\left(\frac{l^2 + a^2 m_\pi^2 - 1}{\alpha}\right)^{1/2}\,. \qquad \text{(A.22)}$$

Evaluating this expression for $s = 1$ one finds

$$a^2 H\big|_{s=1} = -\sum_{l=3}^{\infty}\frac{\pi a^2 l}{2\alpha P}\frac{1}{[1+(-1)^l e^{2\pi P}]}\,. \qquad \text{(A.23)}$$

Whilst the derivative at $s = 0$ is

$$\frac{d}{ds}H\bigg|_{s=0} = -\sum_{l=3}^{\infty} 2l\ln\left[1+(-1)^l e^{-2\pi P}\right]\,. \qquad \text{(A.24)}$$

The sums in the last two expressions are convergent and may be evaluated numerically. The last expression clearly displays the connection to a theory at finite temperature for $\alpha > 0$. If we interpret $\sqrt{\alpha}$ as a temperature, the above expression reduces to the finite temperature free energy associated to a field theory on a two sphere, along with certain parity reversal operations.

B. A check of the zeta function

In [20] a series expansion in α is given for the zeta function on a squashed three-sphere for the value of the mass $m_\pi^2 = R/6$ (note that this is not the conformal value in three dimensions). The result is

$$\zeta(s) = (a\mu)^{2s}\Big[\zeta_R(2s-1) + \frac{\alpha}{3}s\zeta_R(2s-2) + 4\alpha^2 \times$$
$$\times\left(\frac{1}{12}s\zeta_R(2s) + 2s(s+1)\left[\frac{1}{80}\zeta_R(2s-2) - \frac{1}{36}\zeta_R(2s) + \frac{1}{45}\zeta_R(2s+2)\right]\right)\Big] + \cdots\,.$$

We have checked our expressions against this expansion and found very good agreement at each order in α. The agreement is rather nontrivial as it involves very precise cancellations between the terms involving F and G in the zeta function (A.4). It does not however provide a check of the H term (A.21) as this depends nonperturbatively on α. However we have derived this remaining term several times and furthermore it agrees precisely with the contribution calculated in [23].

C. The zeta function at large squashing

The limit $\alpha \to \infty$ of the zeta function may be treated directly because one of the sums may be turned into an integral in this limit. The zeta function sum (3.5) may be rewritten as

$$\zeta(s) = \frac{(a\mu)^{2s}}{\alpha^s} \sum_{k=-\infty}^{\infty} \sum_{l \in |k|+1+2\mathbb{N}} \frac{l}{\left(\frac{l^2-1}{\alpha} + k^2 + \frac{a^2 m_\pi^2}{\alpha}\right)^s} . \tag{C.1}$$

In the limit $\alpha \to \infty$, we may define $x = l/\alpha^{1/2}$ and let

$$\sum_{l \in |k|+1+2\mathbb{N}} \quad \to \quad \frac{\alpha^{1/2}}{2} \int_{(|k|+1)/\alpha^{1/2}}^{\infty} dx\,. \tag{C.2}$$

This treatment is only allowed for the $k = 0$ terms if $a^2 m_\pi^2/\alpha$ remains finite as $\alpha \to \infty$. We will see that indeed this is the case at strong coupling, $a\lambda \to \infty$. Doing the x integral, the zeta function becomes

$$\zeta(s) = \frac{(a\mu)^{2s}}{4\alpha^{s-1}} \frac{1}{(s-1)} \sum_{k=-\infty}^{\infty} \frac{1}{\left(k^2 + \frac{a^2 m_\pi^2}{\alpha}\right)^{s-1}} \qquad \text{as} \qquad \alpha \to \infty\,. \tag{C.3}$$

To find the behaviour of $a^2 m_\pi^2$ at strong coupling and as $\alpha \to \infty$ we need to solve (3.11), that is $\zeta(1) = 0$. We may evaluate $\zeta(1)$ by analytic continuation of (C.3) using the techniques of appendix A. The result is

$$\zeta(1) = -\frac{a^2}{2} \left[\frac{\pi a m_\pi}{\alpha^{1/2}} + \ln\left(1 - e^{-2\pi a m_\pi/\alpha^{1/2}}\right) \right] \qquad \text{as} \qquad \alpha \to \infty\,. \tag{C.4}$$

Setting this expression to zero one obtains equation (3.12) in text

$$a^2 m_\pi^2 = \frac{\alpha}{\pi^2} \ln^2 \left[\frac{1+\sqrt{5}}{2} \right] \qquad \text{as} \qquad \alpha \to \infty\,. \tag{C.5}$$

To compute the free energy in this limit, we need to evaluate $\zeta'(0)$. By the usual methods we obtain

$$\zeta'(0) = \frac{\alpha}{2} \left[\frac{2\pi (a m_\pi)^3}{3\alpha^{3/2}} + \pi \int_{a m_\pi/\alpha}^{\infty} \frac{2(t^2 - a^2 m_\pi^2/\alpha)}{e^{2\pi t} - 1} \right] \qquad \text{as} \qquad \alpha \to \infty\,. \tag{C.6}$$

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The integral may be expressed in terms of dilogarithms and trilogarithms. However, for the particular value of the mass given by (C.5) the expression simplifies dramatically to give the strong coupling result

$$\zeta'(0) = \frac{\zeta_R(3)}{5\pi^2}\alpha \qquad \text{as} \qquad \alpha \to \infty\,, \tag{C.7}$$

which immediately leads to equation (3.14) in the text. We have checked numerically that these strong coupling results actually match our analytically continued expressions involving F, G and H in (A.4) in the large α regime. The matchings rely on delicate cancellations between the F, G and H terms and thus provide a nontrivial check of our analytic continuations. The weak coupling result (3.13) is obtained similarly, but in this case $a^2m_\pi^2/\alpha \to 0$ as $\alpha \to \infty$ and one has to treat the $k = 0$ sum separately.

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